§16.3 Finding potential functions for conservative vector fields.

Summary of what we have:

- $\vec{F} = \overrightarrow{(M, N, P)}$ is <u>conservative</u> if $\vec{F} = \nabla f$ for some f.

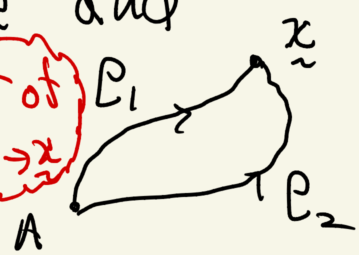
  Q1: How do we know a given f is conservative?

  Q2: Given f is conservative, how do we find f?

- <u>Theorem ①</u>: If $\oint_C \vec{F} \cdot ds = 0$ for every closed curve, then $\vec{F}$ is conservative and

$$f(\underset{\sim}{x}) = \int_A^{\underset{\sim}{x}} \vec{F} \cdot \vec{T}\, ds$$

indep't of path $A \to \underset{\sim}{x}$



- <u>Theorem ②</u>: If $\text{Curl}\,\vec{F} = 0$ in a simply connected Domain D, then $\vec{F}$ is conservative

<u>Picture</u>: $\mathbb{R} \xrightarrow{\nabla} \mathbb{R}^3 \xrightarrow{\text{Curl}} \mathbb{R}^3 \xrightarrow{\text{Div}} \mathbb{R}$

2 in a row give zero: $\text{Curl}(\nabla f) = 0 = \text{Div}(\text{Curl}\,\vec{F})$

Idea: Stokes Thm: $\iint_S \text{Curl}\,\vec{F} \cdot \vec{n}\, dS = \int_C \vec{F} \cdot \vec{T}\, ds$

"<u>Simply connected</u> means you can contract closed curves to a point, so Stokes $\Rightarrow \oint_C \vec{F} \cdot \vec{T}\, ds = 0 \; \forall$ closed $C \Rightarrow \vec{F}$ conservative"

Theorems 1 & 2 answer Q1 —

Q1: How do you determine whether $\vec{F}$ is conservative?

Before addressing Q2, we do an example:

Example 1 Its easy to construct conservative vector fields - just choose f & set $\vec{F} = \nabla f$

Eg choose $f(\underset{\sim}{x}) = f(x,y,z) = xy^2z^3 + y^2z + 2z$

Set $\vec{F} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$\Rightarrow \vec{F} = \left(y^2z^3, 2xyz^3 + 2yz, 3xy^2z^2 + y^2 + 2\right)$

$\frac{\partial f}{\partial x} = y^2z^3$

$\frac{\partial f}{\partial y} = 2xyz^3 + 2yz$

$\frac{\partial f}{\partial z} = 3xy^2z^2 + y^2 + 2$

But given $\vec{F}$ not knowing f, how do you know f exists such that $\vec{F} = \nabla f$?

Ans: Thm 2 says check $\text{Curl}\,\vec{F} = 0$ and make sure this hold on a simply connected Domain D. Since $\vec{F}$ is defined for all $\underset{\sim}{x} \in \mathbb{R}^3$, (clearly $\mathbb{R}^3$ is simply connected), we need only check that $\text{Curl}\,\vec{F} = 0$ for every $\underset{\sim}{x} \in \mathbb{R}^3$. We already know this since $\text{Curl}\,\nabla f = 0$, but assume we don't know f!

Example 1 (cont) so lets assume we are given $\vec{F} = \overrightarrow{(y^2z^3,\ 2xyz^3 + 2yz,\ 3xy^2z^2 + y^2 + 2)}$ and we do not know it come from $f(\underset{\sim}{x}) = xy^2z^3 + y^2z + 2z$

How would we determine $\vec{F}$ is conservative?

Ans: We take the Curl:

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ y^2z^3, & 2xyz^3 + 2yz, & 3xy^2z^2 + y^2 + 2 \end{vmatrix} \begin{array}{l} \leftarrow \nabla = \overrightarrow{(\partial_x, \partial_y, \partial_z)} \\ \\ \leftarrow \vec{F} = \overrightarrow{(M, N, P)} \end{array}$$

$$= \underset{\sim}{i}\left(P_y - N_z\right) - \underset{\sim}{j}\left(P_x - M_z\right) + \underset{\sim}{k}\left(N_x - M_y\right)$$

$$= \underset{\sim}{i}\,\underbrace{\left(6xyz^2 + 2y - 6xyz^2 - 2y\right)}_{0} - \underset{\sim}{j}\,\underbrace{\left(3y^2z^2 - 3y^2z^2\right)}_{0} + \underset{\sim}{k}\,\underbrace{\left(2yz^3 - 2yz^3\right)}_{0} = 0$$

Conclude: $\text{Curl}\,\vec{F} = 0$ on a simply connected domain $D = \mathbb{R}^3 \underset{\text{Thm 2}}{\Longrightarrow} \vec{F}$ is conservative.

Q2: How do we find $f(x,y,z)$ given we know $\vec{F}$ is conservative?

We now describe the procedure which always recovers f such that $\nabla f = \vec{F}$ when we are given $\vec{F}$ and we know $\vec{F}$ is conservative. We work out the procedure in this example — the general will then be clear from the steps in the example

$$\vec{F} = (\underbrace{y^2 z^3}_{M}, \underbrace{2xyz^3 + 2yz}_{N}, \underbrace{3xy^2z^2 + y^2 + 2}_{P})$$

Goal — recover f:

① Since $\frac{\partial f}{\partial x} = M$, write $f = \underbrace{\int_x M\,dx}_{\text{makes } \frac{\partial f}{\partial x} = M}$

$$f = \int_x M\,dx = \int_x y^2 z^3\,dx$$

$$= xy^2z^3 + g(y,z)$$

General term that vanishes when you take $\frac{\partial f}{\partial x}$

② Take $\frac{\partial}{\partial y}$ of f in ① and compare it with N

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\{xy^2z^3 + g(y,z)\} = \cancel{2xyz^3} + \frac{\partial g}{\partial y}(y,z)$$

$$= \underbrace{\cancel{2xyz^3} + 2yz}_{N}$$

It remains to find $g(y,z)$

3 The equation for $g(y,z)$ is

$$\frac{\partial}{\partial y} g(y,z) = 2yz$$

So $g(y,z) = \int_y 2yz\,dy = y^2 z + h(z)$

Now put this new info about g back into 1

$$f = xy^2z^3 + y^2 z + h(z) \qquad (*)$$

4 Now do step 2 with z in place of y using the updated f

$$\frac{\partial f}{\partial z} = 3xy^2z^2 + y^2 + h'(z) = \underbrace{3xy^2z^2 + y^2 + 2}_{P}$$

5 The equation for h is

$$h'(z) = 2$$

So $h(z) = 2z + \text{const}$

Use this to update $(*)$ to obtain f

$$f(x,y,z) = \underbrace{xy^2z^3 + y^2z + 2z}_{\text{our original } f} + \underbrace{\text{const}}_{\nabla(f+C) = \nabla f = \vec{F}}$$

Example ② $\vec{F} = \Big(\underbrace{\frac{\cos x}{r} - \frac{x\sin x}{r^2}}_{M}, \underbrace{\frac{y\sin x}{r^2}}_{N}, \underbrace{\frac{z\sin x}{r^2}}_{P}\Big)$ $\quad r = \sqrt{x^2+y^2+z^2}$

Determine whether $\vec{F}$ is conservative w/o finding f.

Soln: $\vec{F}$ is defined for all $\underset{\sim}{x} \in \mathbb{R}^3$ except $\underset{\sim}{x} = 0$, which is simply connected. Thus by Thm ②, it suffices to check $\text{Curl}\,\vec{F} = 0$ for all $\underset{\sim}{x} \neq 0$.

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \underset{\sim}{i}\,(P_y - N_z) - \underset{\sim}{j}\,(P_x - M_z) + \underset{\sim}{k}\,(N_x - M_y)$$

$$P_y = \frac{\partial}{\partial y}\left(\frac{z\sin x}{r^2}\right) = z\sin x\,(-2)r^{-3}\frac{y}{r} \;;\; N_z = \frac{\partial}{\partial z}\left(\frac{y\sin x}{r^2}\right) = y\sin x\,(-2)r^{-3}\frac{z}{r}$$

$$P_y - N_z = -\frac{2yz\sin x}{r^4} + \frac{2zy\sin x}{r^4} = 0 \quad \checkmark$$

Similarly - $P_x - M_z = 0 = N_x - M_y$ (Homework)

Conclude: $\text{Curl}\,\vec{F} = 0$ in S.C. Domain $\underset{\text{Thm②}}{\Rightarrow}$ Conservative

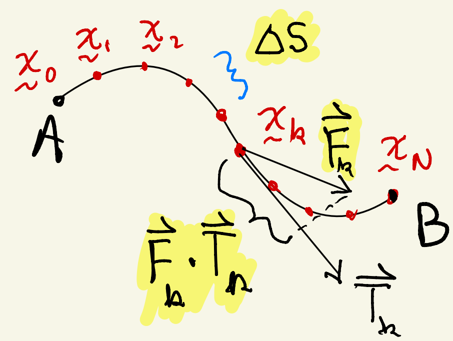
(7)

Example (3) List and explain properties of line integrals

$\int_C \vec{F}\cdot\vec{T}\,ds$ is defined as a Riemann Sum

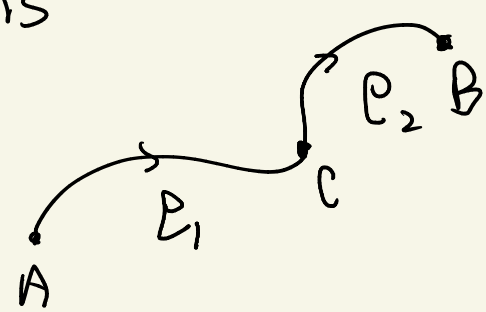
$$\int_C \vec{F}\cdot\vec{T}\,ds = \lim_{N\to\infty} \sum_{k=1}^{N} \vec{F}_k\cdot\vec{T}_k\,\Delta s$$

Since this is just a sum, and it doesn't matter what order you add things up, you get

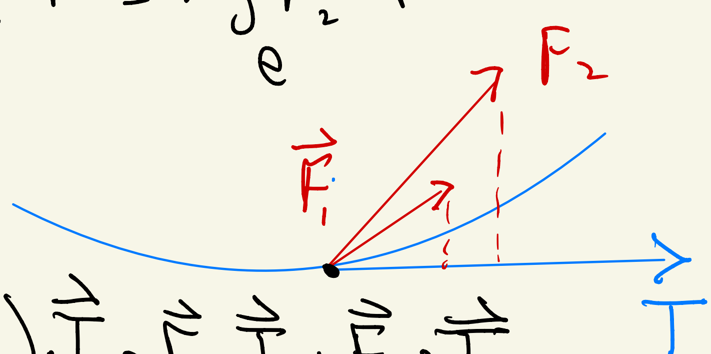


(1) $\int_C \vec{F}\cdot\vec{T}\,ds = \int_{C_1} \vec{F}\cdot\vec{T}\,ds + \int_{C_2} \vec{F}\cdot\vec{T}\,ds$

$C = C_1 + C_2$



(2) $\int_C (\vec{F}_1+\vec{F}_2)\cdot\vec{T}\,ds = \int_C \vec{F}_1\cdot\vec{T}\,ds + \int_C \vec{F}_2\cdot\vec{T}\,ds$



$(\vec{F}_1+\vec{F}_2)\cdot\vec{T} = \vec{F}_1\cdot\vec{T} + \vec{F}_2\cdot\vec{T}$

(3) $\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = -\int_{-\mathcal{C}} \vec{F}\cdot\vec{T}\,ds$

The only change in the arclength parameter picture is $\vec{T}$ reverses its sign...

To see this note:

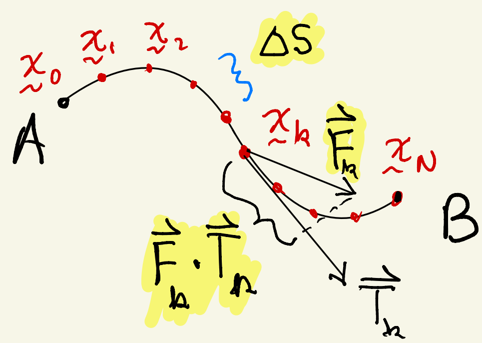
$$\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = \lim_{N\to\infty} \sum_{k=1}^{N} \vec{F}_k\cdot\vec{T}_k\,\Delta s$$

using the $\vec{T}_k$ from the curve $\mathcal{C}$

$$\int_{-\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = \lim_{N\to\infty} \sum_{k=1}^{N} \vec{F}_k\cdot(-\vec{T}_k)\,\Delta s$$

$$= -\lim_{N\to\infty} \sum_{k=1}^{N} \vec{F}_k\cdot\vec{T}_k\,\Delta s = -\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds$$

$$\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = \lim_{N\to\infty} \sum_{k=1}^{N} \vec{F}_k\cdot\vec{T}_k\,\Delta s$$



For $-\mathcal{C}$, the x_k's run from $B \to A$ as $k = 1 \to N$, so $\vec{T}$ reverses its sign...

Famous Example: Recall $\vec{F} = -\frac{y}{r^2}\,\underset{\sim}{i} + \frac{x}{r^2}\,\underset{\sim}{j}$.

This we showed was Curl free:

$$\text{Curl}\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ -\frac{y}{r^2} & \frac{x}{r^2} & 0 \end{vmatrix} = \underset{\sim}{k}\left(\frac{\partial}{\partial y}\left(-\frac{y}{r^2}\right) - \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right)\right)$$

$$\frac{\partial}{\partial y}\left(-\frac{y}{r^2}\right) = -\frac{1}{r^2} + \frac{2y}{r^3}\frac{y}{r}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) = -\frac{1}{r^2} + \frac{2x}{r^3}\frac{x}{r}$$

$$\text{Curl}\vec{F} = -\frac{2}{r^2} + \frac{2(y^2+x^2)}{r^3} = 0 \checkmark$$

But: $\vec{F}$ not defined at $r=0$, any $z \Rightarrow$ not defined on z-axis, so we do not have $\text{Curl}\vec{F} = 0$ on a simply connected domain $\Rightarrow \oint_C \vec{F}\cdot\vec{T}$ need not always be zero $\Rightarrow \vec{F}$ need not be conservative.

In fact we showed:

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \int_0^{2\pi}(-y,x)(-\sin t,\cos t)\,dt = 2\pi \neq 0$$

$$\vec{r}(t) = \cos t\,\underset{\sim}{i} + \sin t\,\underset{\sim}{j}$$

So there can be no f st $\nabla f = \vec{F}$, o.w.,

$$\oint_C \vec{F}\cdot\vec{T}\,ds = 0 \neq 2\pi \;!$$

But there is an interesting continuation of the story...

Consider: $\vec{F} = -\frac{y}{r^2}\,\underset{\sim}{i} + \frac{x}{r^2}\,\underset{\sim}{j}$

with C_n given by: $\vec{r}(t) = \cos t\,\underset{\sim}{i} + \sin t\,\underset{\sim}{j} + t(2n\pi - t)\,\underset{\sim}{k}$

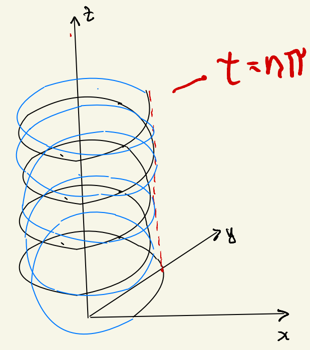
$$0 \le t \le 2n\pi$$

Note C_n is closed: $\vec{r}(0) = \cos 0\,\underset{\sim}{i} + \sin 0\,\underset{\sim}{j} = \underset{\sim}{i}$

$\vec{r}(2n\pi) = \cos 2n\pi\,\underset{\sim}{i} + \sin 2n\pi\,\underset{\sim}{j} = \underset{\sim}{i}$

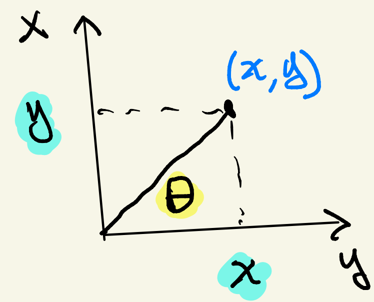
C_n spirals around z-axis n-times.

$$\oint_{C_n} \vec{F}\cdot\vec{T}\,ds = \int_0^{2n\pi}(-y,x)(-\sin t,\cos t)\,dt = 2n\pi$$



Consider now the "angle function"

$$\theta = \arctan\left(\frac{y}{x}\right)$$



$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{x^2}{x^2+y^2}\frac{y}{-x^2}$$

$$= -\frac{y}{r^2}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \frac{x^2}{x^2+y^2}\frac{1}{x} = \frac{x}{r^2}$$

So $\nabla\theta(x,y) = \overbrace{\left(-\frac{y}{r^2}, \frac{x}{r^2}\right)} = \vec{F}$

Thus $\oint_{C_n} \vec{F}\cdot\vec{T}\, ds =$ "$\theta(B)-\theta(A)$" $= 2\pi n$ ✓

But this is not really correct because

$\theta = \arctan\left(\frac{y}{x}\right)$ is not defined on x-axis $\Rightarrow$

θ not defined when C_n crosses $x=0$!

Still: Something looks correct !

In fact: This is the central issue of Complex Variables – How to put the $i=\sqrt{-1}$ into Calc

Consider $f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{r^2} + \frac{-y}{r^2} i$

$$\oint_{C_n} \frac{dz}{z} = \oint_{C_n} \frac{dx+idy}{x+iy} = \oint_{C_n} \frac{(dx+idy)(x-iy)}{x^2+y^2}$$

$$= \oint_{C_n} \frac{x\,dx + y\,dy}{r^2} + i\oint_{C_n} \frac{-y\,dx + x\,dy}{r^2} \qquad \vec{r} = \cos t\, \underset{\sim}{i} + \sin t\, \underset{\sim}{j}$$

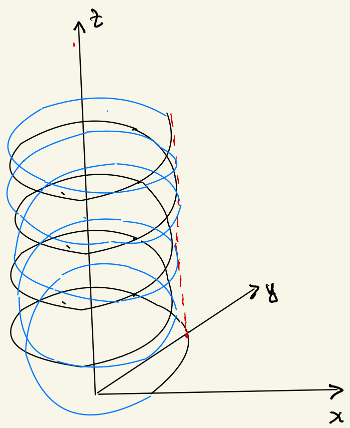
$$= \int_0^{2\pi n} \cancel{\frac{\cos t(-\sin t) + \sin t(\cos t)}{r^2}\, dt} + i\oint_{C_n} \vec{F}\cdot\vec{T}\, ds = \boxed{2\pi n i}$$

0

Turns out: You can make sence of $f(z) = z^n$, z^{-n}, and we can differentiate and integrate, and

$\int_{C_n} z^n dz = 0$ for every n <u>except</u> $n=-1$

Turns out: $f(z) = \frac{1}{z}$, $\oint_{C_n} \frac{dz}{z} = 2\pi n i$, is the most important function in complex variables!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